Some Generalizations of Pareto Distribution with Applications

Gokarna R. Aryal

Department of Mathematics, Statistics and Computer Science
Purdue University Northwest
Hammond, IN, USA
E-mail: aryalg@pnw.edu

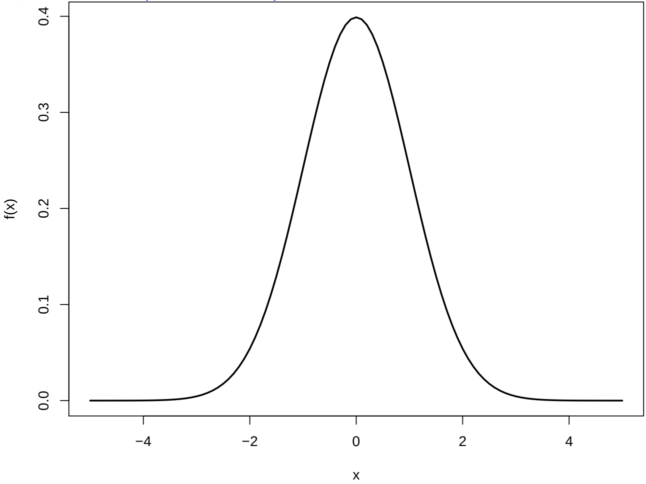
May 12, 2018

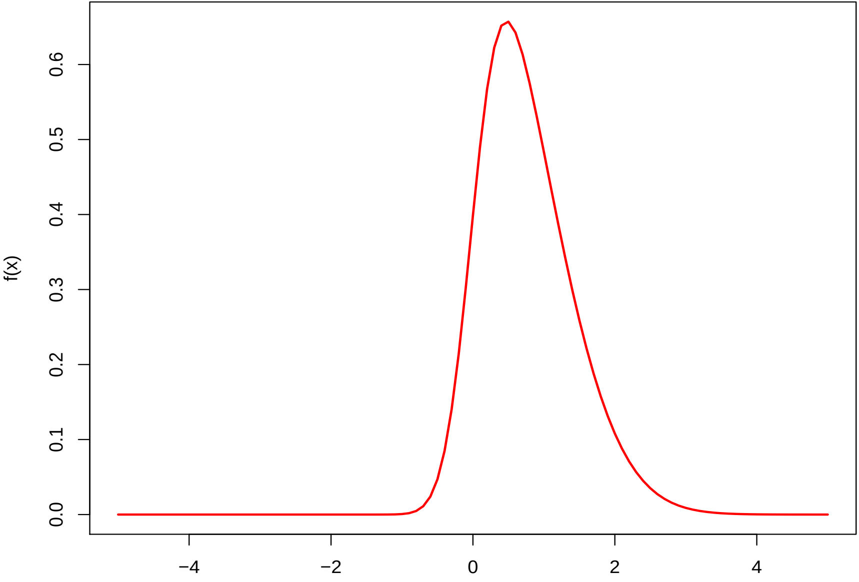
Outline

- **What are Generalized Distributions?**
- **Key Ingredients for Generalization**
- **Generalization of Pareto Distribution**
- **Beta Exponential Pareto Distribution**
- **Assessing the Effectiveness of BEP Distributions**
- **Some Applications**

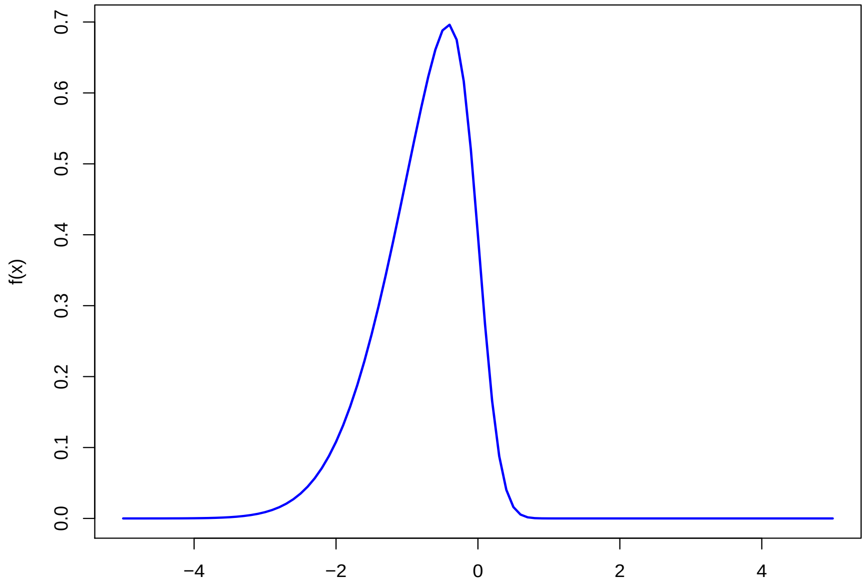
Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

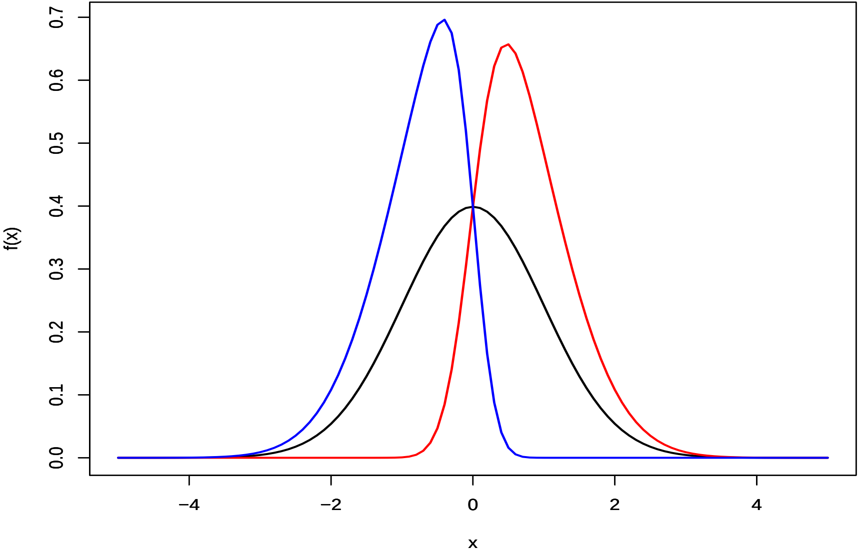




f(x)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−4
−2
0
2
4


f(x)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
−4
−2
0
2
4

Skew Normal Distribution



Motivation of the Study

- Statistical analysis depends on the assumed probability model or distributions
- The nature of data may guide the choice of which distribution among a set of parametric distributions
- There are many important problems where the data don't follow any of the standard models
- The standard distributions are more exceptions rather than the reality
- Several ways of adding one or more parameters to a distribution
- Not all extension are applicable to all models.
- We will present some recent generalization techniques

Motivation of the Study

Adding one or more parameters to a distribution makes it more flexible

- Produce skewness for symmetrical models;
- Define special models with different shapes of hazard rate function;
- Construct heavy-tailed distributions for modeling various real data sets;
- Make the kurtosis more flexible compared to that of the baseline distribution;
- Generate distributions which are left-skewed, right-skewed, symmetric, J-shaped or reversed-J shape;

Generalization Method

Commonly used generalizers: Let $G(x)$ be the cdf of a random variable

- Skew -Symmetric distribution (Azzalini, 1985)

$$f(x) = 2g(x)G(\lambda x) \qquad -\infty < x < \infty, \quad -\infty < \lambda < \infty$$

- Exponentiated-G distribution (Mudholkar & Srivastava, 1993)

  $F_{EG}(x) = [G(x)]^{\nu}\,, \qquad \nu > 0$

- The Marshall Olkin-G family (Marshall and Olkin, 1997)

$$F(x;\delta,\psi) = 1 - \frac{\delta \overline{H}\,(x;\psi)}{1-(1-\delta)\,\overline{H}\,(x;\psi)},\ x \in \mathbb{R},\ \delta > 0.$$

  where, $H(x;a,\psi) = 1 - [\overline{G}(x;\psi)]^{a}$

- Beta-G distribution (Eugene, Lee & Famoye, 2002)

$$F_{BG}(x) = \frac{B_{G(x)}(a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt, \quad a > 0, b > 0$$

Generalization Methods- Contd.

- Transmuted-G distribution (Aryal & Tsokos, 2009)

  $$F_{TG}(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \qquad |\lambda| \leq 1$$

- Kumaraswamy-G distribution (Cordeiro & de Castro, 2011)

  $$F_{KG}(x) = 1 - [1 - G(x)^c]^d, \qquad c > 0, d > 0$$

- Zero truncated Poisson (ZTP) distribution to develop generalizers, (Ramos *et al.*, 2015)

$$P(N = n) = \frac{1}{[1 - \exp(-\lambda)]} \frac{\exp(-\lambda)\lambda^n}{n!} \text{ for } n = 1, 2, ...$$

$$F(x) = \frac{1 - \exp[-\lambda G(x; \boldsymbol{\xi})]}{[1 - \exp(-\lambda)]}.$$

- Topp-Leone, (Rezaei *et al.*, 2017)

$$F_{TLG}(x) = \left\{ G(x)^\theta \left[2 - G(x)^\theta \right] \right\}^\alpha, \quad x \geq 0, \theta > 0$$

Transmutation Map [Shaw and Buckley, 2007]

Given two distributions with a common sample space with CDFs, F_1, F_2, the Rank Transmutation Maps are defined as

$$
\begin{aligned}
G_{R_{12}}(u) &= F_2(F_1^{-1}(u)) \\
G_{R_{21}}(u) &= F_1(F_2^{-1}(u))
\end{aligned}
$$

The functions $G_{12}(u)$ and $G_{21}(u)$ both map the unit interval $I = [0, 1]$ into itself.
The quadratic RTM (QRTM), has the following simple quadratic form,

$$G_{12}(u) = u + \lambda u(1 - u)$$

for $|\lambda| \leq 1$.

Transmuted Distribution

The quadratic RTM (QRTM), has the following simple quadratic form,

$$G_{12}(u) = (1 + \lambda)u - \lambda u^2$$

Consider a Random Variable X with CDF $G(x)$

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2$$

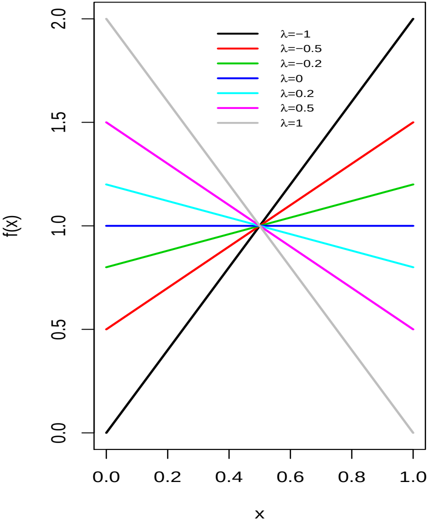
where, $|\lambda| \leq 1$ is called the transmuted parameter.
The corresponding PDF is

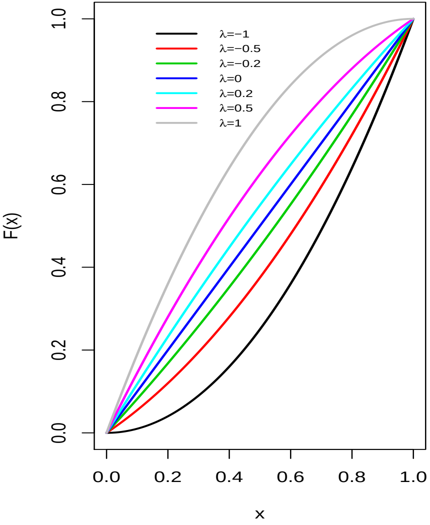
$$f(x) = g(x)[(1 + \lambda) - 2\lambda G(x)]$$

Remark: $\lambda = 0$ gives the distribution of the base random variable.

On the Transmuted Extreme Value Distribution with Application (Aryal and Tsokos, 2009) Nonlinear Analysis: Theory, Methods & Applications

Transmuted Distribution





Transmuted Distributions

Tahir and Cordeiro (2016, Journal of Statistical Distributions and Applications) Fifty-one papers with transmutation map have been published.

Table 1 Contributed work on transmuted distributions

S.No.	Pioneer Year	Distribution	Author(s)
1.	2009	Transmuted extreme value	Aryal and Tsokos (2009)
2.	2011	Transmuted Weibull	Aryal and Tsokos (2011)
			Ahmad et al. (2015b)
			Khan et al. (2016b)
3.	2013	Transmuted log-logistic	Aryal (2013)
			Granzotto and Louzada (2015)
4.	2013	Transmuted Rayleigh	Merovci (2013a)
			Ahmad et al. (2015a)
5.	2013	Transmuted exponentiated-exponential	Merovci (2013b)
			Khan et al. (2015a)
6.	2013	Transmuted Fréchet	Mahmoud and Mandouh (2013)
7.	2013	Transmuted Lomax	Ashour and Eltehiwy (2013a)
8.	2013	Transmuted Lindley	Merovci (2013c)
9.	2013	Transmuted quasi-Lindley	Elbatal and Elgarhy (2013)
10.	2013	Transmuted exponentiated-Lomax	Ashour and Eltehiwy (2013b)

Transmuted Pareto Distribution

A random variable X is said to have Pareto distribution if its pdf and cdf are given by

$$\begin{aligned} g(x;\lambda,\theta) &= \frac{\theta\lambda^{\theta}}{x^{\theta+1}}, \\ G(x;\lambda,\theta) &= 1-\left(\frac{\lambda}{x}\right)^{\theta}, \qquad \theta > 0 \end{aligned}$$

where $x \in (\lambda, \infty)$, $\lambda > 0$ is the (threshold) scale parameter.

$$f(x;\lambda,\theta,\rho) = \frac{\theta\lambda^{\theta}}{x^{\theta+1}}\left[1-\rho+2\rho\left(\frac{\lambda}{x}\right)^{\theta}\right],$$

where λ is the (necessarily positive) minimum possible value of X. The corresponding CDF of the transmuted Pareto distribution is given by

$$F(x;\lambda,\theta,\rho) = \left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]\left[1+\rho\left(\frac{\lambda}{x}\right)^{\theta}\right].$$

The distribution becomes the Pareto when $\rho = 0$.

Beta Exponential Pareto Distribution

Exponential Pareto distribution

Exponential distribution: A random variable X is said to have exponential distribution if its pdf and cdf are given by

$$\begin{aligned} f_1(x;\alpha) &= \alpha e^{-\alpha x}, \\ F_1(x;\alpha) &= 1 - e^{-\alpha x}, \qquad x > 0, \alpha > 0 \end{aligned}$$

Pareto distribution: A random variable X is said to have Pareto distribution if its pdf and cdf are given by

$$\begin{aligned} g_1(x;\lambda,\theta) &= \frac{\theta\lambda^{\theta}}{x^{\theta+1}}, \\ G_1(x;\lambda,\theta) &= 1 - \left(\frac{\lambda}{x}\right)^{\theta}, \qquad \theta > 0 \end{aligned}$$

where $x \in (\lambda, \infty)$, $\lambda > 0$ is the (threshold) scale parameter.
Exponential Pareto distribution: Kareema and Boshi (2013) introduced a new distribution that is dependent on both the Exponential distribution and Pareto distribution, called Exponential Pareto distribution. The cdf and pdf of exponential Pareto distribution is given by

$$\begin{aligned} G(x;\alpha,\theta,\lambda) &= 1 - e^{-\alpha(\frac{x}{\lambda})^{\theta}}, x > 0. \\ g(x;\alpha,\theta,\lambda) &= \frac{\theta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\alpha(\frac{x}{\lambda})^{\theta}}, x > 0, \end{aligned}$$

Beta Exponential Pareto (BEP) distribution

Beta-G distribution Let $G(x)$ be the cumulative distribution function (cdf) of a random variable X. Then the cdf of the beta-G random variable is given by

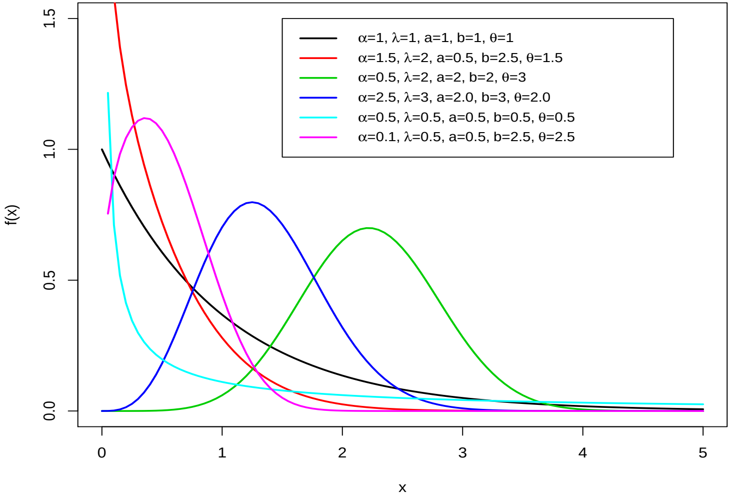
$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw.$$

Beta-Exponetial Pareto Distribution

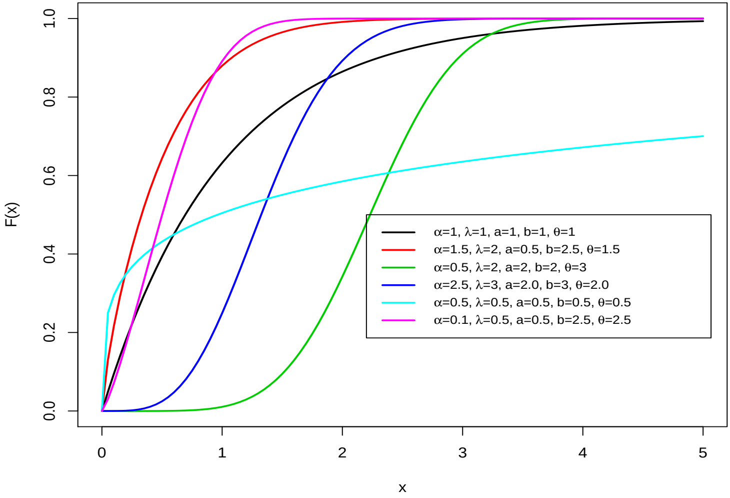
For a random variable X the cdf and pdf of BEP distribution are given by

$$\begin{aligned} F_{BEP}(x) &= \frac{1}{B(a,b)} \int_0^{1-e^{-\alpha(\frac{x}{\lambda})^\theta}} w^{a-1}(1-w)^{b-1} dw \\ f_{BEP}(x) &= \frac{\theta\alpha}{\lambda B(a,b)} \left(\frac{x}{\lambda}\right)^{\theta-1} e^{-b\alpha(\frac{x}{\lambda})^\theta} \left[1 - e^{-\alpha(\frac{x}{\lambda})^\theta}\right]^{a-1}. \end{aligned}$$

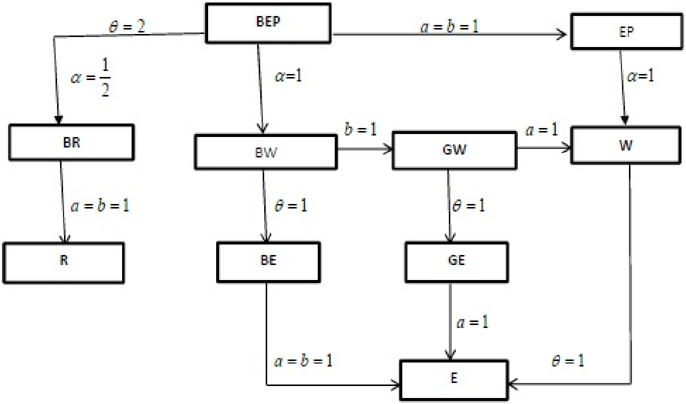
Beta Exponential Pareto distribution-pdf



Beta Exponential Pareto distribution-cdf



Special case of BEP distribution



Sub-models of BEP distribution:

G=Generalized, B=beta, W=Weibull, E=exponential, P=Pareto, R=Rayleigh

Series Representation

Expansion of the the PDF:

$$\begin{aligned} f_{BEP}(x) &= \frac{\theta\alpha}{\lambda B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\alpha\left(\frac{x}{\theta}\right)^{\theta}} \left[1 - e^{-\alpha\left(\frac{x}{\lambda}\right)^{\theta}}\right]^{a+i-1} \\ &= \frac{\alpha\theta}{\lambda B(a,b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \binom{a+i-1}{j} \left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(j+1)\alpha\left(\frac{x}{\lambda}\right)^{\theta}} \\ &= \sum_{j=0}^{\infty} w_j g(x;\theta,\lambda,\alpha(j+1)) \end{aligned}$$

where

$$w_j = \frac{1}{(j+1)B(a,b)} \sum_{i=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \binom{a+i-1}{j}$$

$g(x;\theta,\lambda,\alpha(j+1))$ denotes the exponential Pareto distribution with parameters θ, λ, and $\alpha(j+1)$.

The quantiles are given by:

$$x_q = \lambda \left[-\frac{1}{\alpha} \log\left(1 - I_q^{-1}(a,b)\right) \right]^{1/\theta}.$$

where $I_q^{-1}(a,b)$ is the inverse of the incomplete beta function with parameters a and b.

Random number generator

Let $u \sim U(0,1)$. Solving the expression $F(x) = u$ gives

$$x = \lambda \left[-\frac{1}{\alpha} \log\left(1 - I_u^{-1}(a,b)\right) \right]^{1/\theta}, \qquad 0 < u < 1$$

Further, we can use quantiles to obtain the median, as well as octiles and then the measure of Bowley skewness and Moors kurtosis.

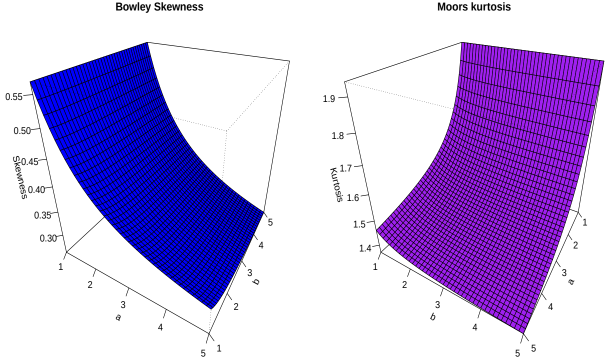
Quantile based skewness and kurtosis

- Bowley skewness (B_{sk}):

$$B_{sk} = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1} = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}.$$

- Moors kurtosis (M_{ku}):

$$M_{ku} = \frac{(E_7 - E_5) + (E_3 - E_1)}{E_6 - E_2} = \frac{Q_{0.875} - Q_{0.625} + Q_{0.375} - Q_{0.125}}{Q_{0.75} - Q_{0.25}},$$



Parameter Estimation

Log-likelihood function:

$$
\begin{aligned}
\ell &= n\ln(\theta) + n\ln(\alpha) - n\theta\ln(\lambda) + n\ln\left(\Gamma(a+b)\right) - n\ln\left(\Gamma(a)\right) - n\ln\left(\Gamma(b)\right) \\
&\quad +(\theta-1)\sum_{i=1}^{n}\ln(x_i) - b\alpha\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta} + (a-1)\sum_{i=1}^{n}\left[1-e^{-\alpha\left(\frac{x_i}{\lambda}\right)^{\theta}}\right].
\end{aligned}
$$

Normal Equations

$$
\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - b\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta} + (a-1)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\frac{e^{-\alpha\left(\frac{x_i}{\lambda}\right)^{\theta}}}{\left[1-e^{-\alpha\left(\frac{x_i}{\lambda}\right)^{\theta}}\right]},
$$

$$
\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - n\ln(\lambda) + \sum_{i=1}^{n}\ln(x_i) - b\alpha\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\ln\left(\frac{x_i}{\lambda}\right) + (a-1)\alpha\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\ln\left(\frac{x_i}{\lambda}\right)
$$

$$
\frac{\partial \ell}{\partial \lambda} = -\frac{n\theta}{\lambda} + \frac{b\alpha\theta}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta} - \frac{(a-1)\alpha\theta}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\frac{e^{-\alpha\left(\frac{x_i}{\lambda}\right)^{\theta}}}{\left[1-e^{-\alpha\left(\frac{x_i}{\lambda}\right)^{\theta}}\right]}
$$

$$
\frac{\partial \ell}{\partial a} = n\left[\psi(a+b)-\psi(a)\right] + \sum_{i=1}^{n}\left[1-e^{-\alpha\left(\frac{x_i}{\lambda}\right)^{\theta}}\right],
$$

$$
\frac{\partial \ell}{\partial b} = n\left[\psi(a+b)-\psi(b)\right] - \alpha\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}.
$$

Simulation and Inference

One would simulate X by

$$X = \lambda \left[-\frac{1}{\alpha} \log \left(1 - I_u^{-1}(a, b) \right) \right]^{1/\theta}$$

where $U \sim U(0, 1)$ is a uniform random number.

Table: Empirical means, the RMSEs and MAEs of the BEP distribution using 1000 simulations for $a = 2$, $b = 1$, $\lambda = 2$, $\theta = 1$ and $\alpha = 2$

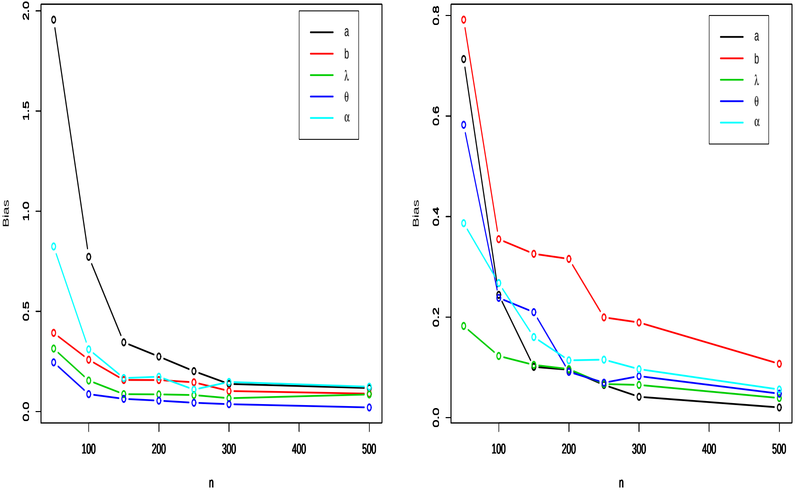
n	Error	$\widehat{a}$	$\widehat{b}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\alpha}$
50		3.956	1.3926	2.3144	1.2454	2.8239
	RMSE	14.6592	2.4814	2.8747	0.7339	6.1358
	MAE	2.8249	0.9608	1.3291	0.4007	1.5715
100		2.7723	1.2587	2.1548	1.0868	2.3108
	RMSE	3.3949	1.7036	2.4368	0.3086	1.9935
	MAE	1.3947	0.7642	1.0490	0.2368	0.8740
150		2.3455	1.1575	2.0867	1.0633	2.1670
	RMSE	1.5215	1.1606	1.6414	0.2589	0.9833
	MAE	0.8934	0.5884	0.8418	0.2000	0.6619
200		2.2744	1.1571	2.0857	1.0543	2.1737
	RMSE	1.5940	1.2265	1.3668	0.2329	1.0121
	MAE	0.7657	0.5685	0.7515	0.1776	0.6073
250		2.2015	1.1459	2.0822	1.0438	2.1094
	RMSE	0.9836	1.0622	1.4900	0.2015	0.8625
	MAE	0.6636	0.5343	0.7380	0.1601	0.5792
300		2.1377	1.1024	2.0664	1.0367	2.1476
	RMSE	0.8095	0.8715	1.2447	0.1765	0.9314
	MAE	0.5556	0.4898	0.6609	0.1396	0.5405
500		2.1173	1.0890	2.0853	1.0204	2.1238
	RMSE	0.6539	0.9087	1.3261	0.1498	0.7142

Simulation

Table: Empirical means , the RMSEs and MAEs of the BEP distribution for $a = 1$, $b = 1$, $\lambda = 1.5$, $\theta = 4.0$ and $\alpha = 1$

n	Error	$\widehat{a}$	$\widehat{b}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\alpha}$
50		1.7133	1.7917	1.6824	4.5826	1.3868
	RMSE	2.4905	3.7096	1.6637	2.5761	2.0975
	MAE	1.0759	1.1633	0.4359	1.6108	0.7583
100		1.2442	1.3550	1.6227	4.2383	1.2677
	RMSE	1.1434	1.6844	0.8743	1.3292	1.3755
	MAE	0.5287	0.6527	0.2874	0.9564	0.5241
150		1.1010	1.3259	1.6049	4.2098	1.1604
	RMSE	0.5581	1.2503	0.5243	1.0697	0.7693
	MAE	0.3674	0.5651	0.2235	0.8018	0.3868
200		1.0953	1.3158	1.5966	4.0911	1.1140
	RMSE	0.4641	1.2834	0.5793	0.8738	0.6331
	MAE	0.3072	0.5185	0.1956	0.6665	0.3155
250		1.0651	1.1995	1.5669	4.0695	1.1154
	RMSE	0.3573	0.8211	0.3302	0.7185	0.4902
	MAE	0.2483	0.4015	0.1632	0.5584	0.2789
300		1.0415	1.1893	1.5651	4.0826	1.0965
	RMSE	0.3134	0.7720	0.2970	0.6722	1.0965
	MAE	0.2312	0.3697	0.1480	0.5277	0.2593
500		1.0203	1.1071	1.5390	4.0477	1.0560

Graphical Display of Bias



Application

The data are the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consists of 72 exceedances for the years 1958-1984. These data have been analyzed by many authors.
1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0,1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8,5.3, 9.7, 27.5, 2.5, 27.0.

Min.	Q1	Median	Mean	Q3	Max.	skewness	kurtosis
0.100	2.125	9.500	12.200	20.12	64.00	1.4725	5.8895

Application

Table: Estimated parameters and their standard errors for the Wheaton river data.

model	a	b	θ	α	λ
BEP	0.5482	0.4984	1.2985	0.0332	0.7474
	(0.2923)	(0.7055)	(0.4979)	(0.1188)	(2.6729)
KwTP	4.2684	17.0139	-0.3687	0.2003	0.1
	(1.5669)	(12.6727)	(0.5308)	(0.0609)	–
KwP	2.8553	85.8468	-	0.0528	0.1
	(0.3371)	(60.4213)	-	(0.0185)	–
BTP	3.9118	17.3874	-0.8518	0.1159	0.1
	(1.8159)	(11.7365)	(0.2588)	(0.0509)	–
BP	3.1473	85.7508	-	0.0088	0.1
	(0.4993)	(0.0001)	-	(0.0015)	–
TP	–	–	-0.952	0.3490	0.1
	–	–	(0.089)	(0.072)	–
E*x*P	2.8797	–	-	0.4241	0.1
	(0.4911)	–	–	(0.0463)	–
P	–	–	–	0.2438	0.1
	–	–	–	(0.0287)	–

Model Comparison

The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria) and the HQIC (Hannan-Quinn information criterion) given by

$$
\begin{aligned}
AIC &= -2\ell(\hat{\Theta}) + 2q \\
BIC &= -2\ell(\hat{\Theta}) + q\log(n) \\
HQIC &= -2\ell(\hat{\Theta}) + 2q\log(\log(n)) \\
CAIC &= -2\ell(\hat{\Theta}) + \frac{2qn}{n-q-1}
\end{aligned}
$$

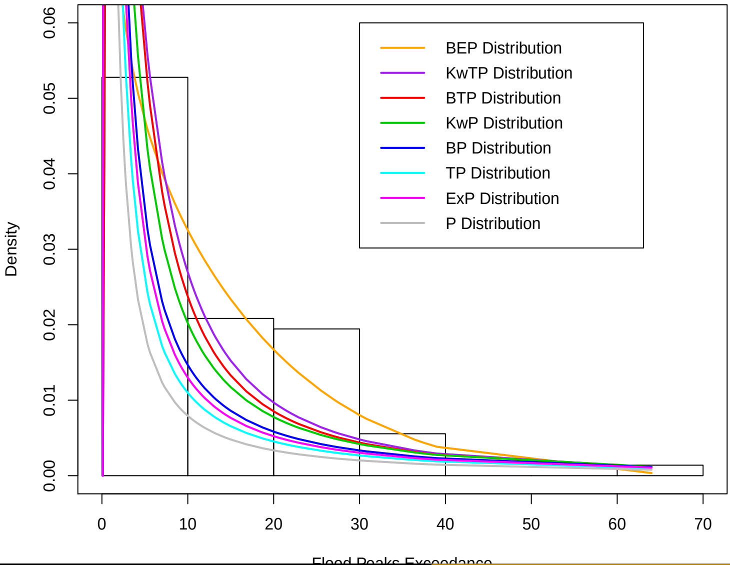
Table: The AIC, CAIC, BIC, HQIC test statistic of the Wheaton river data

Model	statistics				
	$-\ell(.,x)$	AIC	CAIC	BIC	HQIC
BEP	250.979	511.959	512.868	523.342	516.491
*Kw*TP	254.017	516.034	516.641	525.085	519.634
BTP	256.577	521.154	521.760	530.204	524.753
*Kw*P	271.200	548.400	548.753	555.230	551.119
BP	283.700	573.400	573.753	580.230	576.119
TP	286.201	576.402	576.575	580.954	578.214
E*x*P	287.300	578.600	578.774	583.153	580.413
P	303.100	608.200	608.257	610.477	609.106

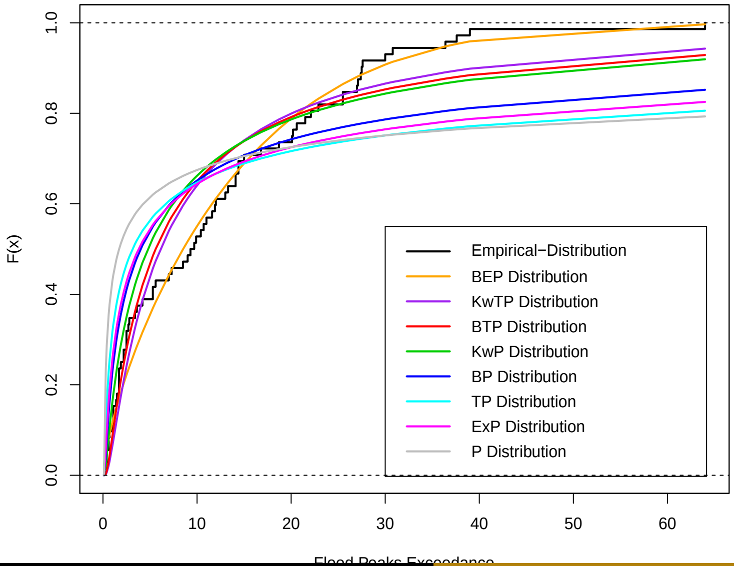
Goodness of fit of the model

Also note that for the subject data the Kolmogorov-Smirnov (KS) test statistic:
$D = 0.10749$ with p-value$= 0.3762$.
Similarly,
Anderson-Darling (A) statistic: 0.6333
Cramér–von Mises (W) statistic: 0.1030
These statistic values also support that the BEP distribution fits well to model exceedances of flood peaks (in m^3/s) of the Wheaton River, Canada.

Model Comparison-PDF



Model Comparison-CDF



Repair times (in hours) for 46 failures of an airborne communications receiver (Lawless).

```
0.20   0.30   0.50   1.50   0.50   0.50   0.60   0.60   0.70   0.70   0.70   0.11
0.80   1.00   1.00   1.00   1.00   1.10   1.30   1.50   1.50   1.50   1.50   2.00
2.00   2.20   2.50   2.70   3.00   3.00   3.30   3.30   4.00   4.00   4.50   4.70
5.00   5.40   5.40   7.00   7.50   8.80   9.00  10.30  22.00  24.50
```

Distribution	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$
BEP	7.0497	0.2545	0.6076	2.6895	0.5438
	(7.4964)	(0.4561)	(15.3900)	(37.2816)	(0.2968)
EP	–	–	2.4017	0.7384	0.8891
	–	–	(162.6504)	(44.4653)	(0.0954)

Model Comparisons:

Distribution	$-\ell$	W	A	D	p-value
BEP	101.2551	0.0232	0.1703	0.0763	0.9516
EP	104.4492	0.0948	0.6467	0.1151	0.5756

Model comparision

One can compute the maximized unrestricted and restricted log-likelihood functions to construct the likelihood ratio (LR) test statistic for testing the models.

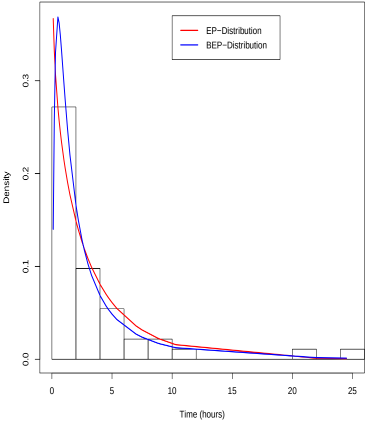
$$H_0 \quad : \quad \text{EP distribution is appropriate}$$
$$H_a \quad : \quad \text{BEP distribution is appropriate}$$

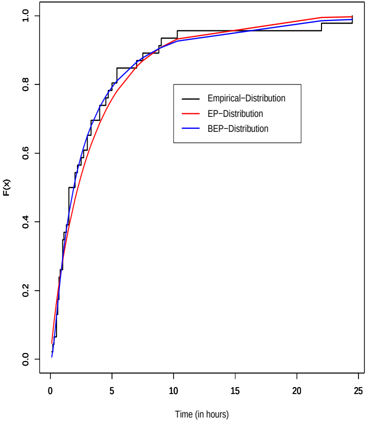
The LR test statistic

$$\omega = 2(\ell(\hat{\varphi}, x) - \ell(\hat{\varphi}_{\mathbf{0}}, x))$$

Note that for the subject data $\omega = 6.3882$ with p-value $= 0.0410$.
It is evident that the BEP fits better than EP distribution for this data set.

Graphical display of the pdf and cdf of BEP Vs. EP models for repair time data





Composite Generalizers

Composite Generalizers-Work in Progress

- Exponentiated-Kumaraswamy
- Exponentiated-Beta
- Exponentiated-Transmuted

A random variable X is said to have a Weibull distribution with parameters α and β if its cdf ($G(x)$) and pdf ($g(x)$) are, respectively, given by

$$\begin{aligned} G(x) &= 1 - \exp\{-(\alpha x)^\beta\} \\ g(x) &= \beta\alpha^\beta x^{\beta-1} \exp\{-(\alpha x)^\beta\}. \end{aligned}$$

- $F_{EKW}(x) = \left[1 - \left\{1 - [1 - \exp\{-(\alpha x)^\beta\}]^c\right\}^d\right]^\nu$
- $F_{EBW}(x) = \left[I_{[1-\exp\{-(\alpha x)^\beta\}]}(a, b)\right]^\nu$
- $F_{ETW}(x) = \left[1 + (\lambda - 1)\exp\{-(\alpha x)^\beta\} - \lambda\exp\{-2(\alpha x)^\beta\}\right]^\nu$

Thank You!