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Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations

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Lump-type solutions, rationally localized in many directions in the space, are analyzed for nonlinear differential equations derived from generalized bilinear differential equations. By symbolic computations with Maple, positive quadratic and quartic polynomial solutions to two classes of generalized bilinear differential equations on f are computed, and thus, lump-type solutions are presented to the corresponding nonlinear differential equations on u, generated from taking a transformation of dependent variables $u = 2(\ln f)_x$.

Keywords: Symbolic computation; generalized bilinear form; positive polynomial solutions; lump and lump-type solutions.

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1. Introduction

The Hirota direct method[1] is one of the most powerful approaches for constructing multi-soliton solutions to integrable nonlinear equations. Its successful idea is to use a transformation of dependent variables to convert nonlinear differential equations into bilinear forms defined in terms of bilinear derivatives called the Hirota bilinear derivatives. Such bilinear forms have been generalized by adopting new rules of taking signs.[2] New broader classes of new nonlinear differential equations derived from generalized bilinear equations have been presented, and their resonant soliton solutions have been constructed by adopting the linear superposition principle.[2,3]

In recent years, there has been growing interest in rationally localized solutions in the space,[4–7] particularly lump solutions, localized in all directions in the space (see, e.g., Refs. 8–12 for typical examples). The KPI equation,

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1}$$

has the following lump solution[13]:

$$u = 4\frac{-[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2\}^2}\,, \tag{2}$$

where a and $b \neq 0$ are two free real constants. More generally, the KPI equation (1) admits the following lump solution[14]:

$$u = 2(\ln f)_{xx} = \frac{4({a_1}^2 + {a_5}^2)f - 8(a_1 g + a_5 h)^2}{f^2}\,, \quad f = g^2 + h^2 + \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)}\,, \tag{3}$$

where

$$g = a_1 x + a_2 y + \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2a_2 a_5 a_6}{{a_1}^2 + {a_5}^2} t + a_4\,,$$

$$h = a_5 x + a_6 y + \frac{2a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2} t + a_8\,,$$

the six parameters a_1, a_2, a_4, a_5, a_6 and a_8 being free real constants satisfying $a_1 a_6 - a_2 a_5 \neq 0$. Rogue wave solutions, which draw big attention from research scientists worldwide, are a particularly interesting class of lump-type solutions,[15,16] and such solutions could be used to describe significant nonlinear wave phenomena in both oceanography[17] and nonlinear optics.[18] There are various discussions on general rational function solutions to integrable equations such as the KdV, KP, Boussinesq and Toda equations.[19–23] It has become a very interesting topic to search for lump solutions or lump-type solutions, rationally localized solutions in many directions in the space, to nonlinear differential equations, based on Hirota bilinear forms and generalized bilinear forms.

Lump solutions to nonlinear differential equations possessing Hirota bilinear forms are analyzed in a recent paper.[24] The basis of success is a set of Hirota bilinear forms and the primary object is a class of multi-variate positive quadratic functions. Necessary and sufficient conditions for the existence of positive quadratic function solutions are presented for general Hirota bilinear equations. Such polynomial solutions yield lump solutions to nonlinear differential equations derived from Hirota bilinear equations under a transformation of either $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$.

In this paper, we would like to consider generalized Hirota bilinear equations, and focus on the two classes of generalized bilinear differential equations involving the prime number $p = 3$ presented in Ref. 2. We will search for their positive quadratic and quartic function solutions by symbolic computation with Maple.

Further, we will present lump-type solutions to the corresponding nonlinear differential equations generated by $u = 2(\ln f)_x$. It is hoped that the study will help us recognize characteristics of nonlinearity more concretely. A few concluding remarks will be given at the end of the paper.

2. Generalized Bilinear Equations

2.1. *Hirota bilinear derivatives and bilinear equations*

Let M be a natural number and $x = (x_1, x_2, \ldots, x_M)^T$ be a column vector of $\mathbb{R}^M$. For f, $g \in C^\infty(\mathbb{R}^M)$, the Hirota bilinear derivatives are defined as follows:

$$D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} f \cdot g := \prod_{i=1}^{M} (\partial_{x_i} - \partial_{x_i'})^{n_i} f(x) g(x')|_{x'=x} \,, \tag{4}$$

where $x' = (x_1', x_2', \ldots, x_M')^T$ and $n_1, \ldots, n_M$ are arbitrary nonnegative integers. For example, we can compute

$$D_i f \cdot g = f_{x_i} g - f g_{x_i} \,,$$
$$D_i D_j f \cdot g = f_{x_i, x_j} g + f g_{x_i, x_j} - f_{x_i} g_{x_j} - f_{x_j} g_{x_i} \,.$$

Assume that $D = (D_1, D_2, \ldots, D_M)$, where each D_i is the first-order Hirota bilinear derivative with respect to x_i.

One important property of the Hirota bilinear derivatives is that

$$D_{i_1} D_{i_2} \cdots D_{i_k} f \cdot g = (-1)^k D_{i_1} D_{i_2} \cdots D_{i_k} g \cdot f \,,$$

where $1 \le i_1, i_2, \ldots, i_k \le M$ need not be distinct. It then follows that if k is odd, we have

$$D_{i_1} D_{i_2} \cdots D_{i_k} f \cdot f = 0 \,.$$

We are interested in the following general Hirota bilinear equation:

$$P(D) f \cdot f = P(D_1, D_2, \ldots, D_M) f \cdot f = 0 \,, \tag{5}$$

where P is a polynomial in M variables. This equation is bilinear indeed. Since the terms of odd powers are all zeros, we can assume that P is an even polynomial, i.e., $P(-x) = P(x)$, while discussing Hirota bilinear equations.

For convenience's sake, we adopt the notation,

$$f_{i_1 i_2 \cdots i_k} = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \,, \quad 1 \le i_1, i_2, \ldots, i_k \le M \,. \tag{6}$$

Therefore,

$$D_i D_j f \cdot f = 2(f_{ij} f - f_i f_j) \,, \quad 1 \le i, \quad j \le M \,, \tag{7}$$

and

$$D_i D_j D_k D_l f \cdot f = 2[f_{ijkl} f - f_{ijk} f_l - f_{ijl} f_k - f_{ikl} f_j - f_{jkl} f_i + f_{ij} f_{kl} + f_{ik} f_{jl} + f_{il} f_{jk}] \,, \quad 1 \le i, j, k, l \le M \,. \tag{8}$$

Motivated by Bell polynomial theories on soliton equations,[25–27] we take one of the transformations:

$$u = 2(\ln f)_{x_1}, \quad u = 2(\ln f)_{x_1 x_1} \tag{9}$$

to formulate nonlinear differential equations from Hirota bilinear equations. All integrable nonlinear equations are such examples.[1,28] For example, for the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0\,,$$

and the KPI and KPII equations,

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0\,, \quad \sigma = \mp 1\,,$$

the transformation $u = 2(\ln f)_{xx}$ provides a link to the (1+1)-dimensional Hirota bilinear form,

$$(D_x D_t + D_x^4) f \cdot f = 0\,, \tag{10}$$

and the (2+1)-dimensional Hirota bilinear form,

$$(D_x D_t + D_x^4 + \sigma D_y^2) f \cdot f = 0\,, \tag{11}$$

respectively.

If a polynomial solution f to a bilinear equation is positive, then the solution u defined by any of the transformations in (9) is analytical and most likely, rationally localized in all directions in the space, and thus, it often presents a lump solution to the corresponding nonlinear differential equation.

2.2. *Generalized bilinear derivatives*

Let $p \in \mathbb{N}$ be given. For $f, g \in C^\infty(\mathbb{R}^M)$, the so-called generalized bilinear derivatives are defined as follows[2]:

$$\begin{aligned}(D_{p,1}^{n_1} D_{p,2}^{n_2} \cdots D_{p,M}^{n_M} f \cdot g)(x) &:= \prod_{i=1}^{M} (\partial_{x_i} + \alpha \partial_{x_i'})^{n_i} f(x) g(x')|_{x'=x} \\ &= \prod_{i=1}^{M} \sum_{j_i=0}^{n_i} \alpha_l^j \binom{n_i}{j_i} \partial_{x_i}^{n_i - j_i} f(x) \partial_{x_i}^{j_i} g(x)\,, \end{aligned} \tag{12}$$

where $x' = (x_1', x_2', \ldots, x_M')^T, n_1, \ldots, n_M$ are arbitrary nonnegative integers, and for any integer m, the mth power of α is defined by

$$\alpha^m = (-1)^{r(m)}$$

with $r(m)$ being the remainder of m divided by p: $m = r(m) \mod p$ with $0 \le r(m) < p$, and thus, $m - r(m) = kp$ for some integer k. It is easy to see that

$$D_{p,i}^n = D_i^n, \quad 1 \le n \le p-1\,,$$

where $1 \le i \le M$ and $p \ge 2$.

When $p = 2k$ ($k \in \mathbb{N}$), the generalized bilinear derivatives reduce to the Hirota bilinear derivatives, since $m - r(m)$ is even and we have $\alpha^m = (-1)^{r(m)} = (-1)^m$. Therefore, $D_{2k,i} = D_i$ for $1 \le i \le M$.

Now, we consider $p = 3$, and then, we have

$$\alpha = -1, \quad \alpha^2 = \alpha^3 = 1, \quad \alpha^4 = -1, \quad \alpha^5 = \alpha^6 = 1, \ldots .$$

It is direct to get

$$\begin{aligned}
D_{3,i} f \cdot g &= D_i f \cdot g = f_i g - f g_i , \\
D_{3,i} D_{3,j} f \cdot g &= D_i D_j f \cdot g = f_{ij} g - f_j g_i - f_i g_j + f g_{ij} , \\
D_{3,i} D_{3,j} D_{3,k} f \cdot g &= f_{ijk} g - f_{ij} g_k - f_{ik} g_j + f_i g_{jk} - f_{jk} g_i + f_j g_{ik} + f_k g_{ij} + f g_{ijk} , \\
D_{3,i} D_{3,j} D_{3,k} D_{3,l} f \cdot g &= f_{ijkl} g - f_{ijk} g_l - f_{ijl} g_k + f_{ij} g_{kl} - f_{ikl} g_j + f_{ik} g_{jl} \\
&\quad + f_{il} g_{jk} + f_i g_{jkl} - f_{jkl} g_i + f_{jk} g_{il} + f_{jl} g_{ik} + f_j g_{ikl} \\
&\quad + f_{kl} g_{ij} + f_k g_{ijl} + f_l g_{ijk} - f g_{ijkl} .
\end{aligned}$$

Therefore, when $g = f$, we obtain

$$\begin{cases}
D_{3,i} f \cdot f = f_i f - f f_i = 0 , \\
D_{3,i} D_{3,j} f \cdot f = 2(f_{ij} f - f_i f_j) , \\
D_{3,i} D_{3,j} D_{3,k} f \cdot f = 2 f_{ijk} f , \\
D_{3,i} D_{3,j} D_{3,k} D_{3,l} f \cdot f = 2(f_{ij} f_{kl} + f_{ik} f_{jl} + f_{il} f_{jk}) .
\end{cases} \tag{13}$$

In particular,

$$D_{3,i}^2 f \cdot f = 2(f_{ii} f - f_i^2), \quad D_{3,i}^3 f \cdot f = 2 f_{iii} f, \quad D_{3,i}^4 f \cdot f = 6 f_{ii}^2 . \tag{14}$$

2.3. *Generalized bilinear equations and polynomial solutions*

Let $P(x)$ be a polynomial in $x \in \mathbb{R}^M$ with degree d_P and $P(0) = 0$ (P may not be even). Suppose that $p \ge 2$ is an integer. Formulate a generalized bilinear equation as follows:

$$P(D_{\langle p \rangle}) f \cdot f = 0 , \tag{15}$$

where $D_{\langle p \rangle} = (D_{p,1}, D_{p,2}, \ldots, D_{p,M})$.

We consider polynomial solutions $f(x)$ with the independent variable $x \in \mathbb{R}^M$. For a monomial $P_k(x) = x_1^{n_1} \cdots x_M^{n_M}$, noting that (i) $\deg(f) = 0$ if and only if $f = \text{const.} \ne 0$ and (ii) $\deg(0) = -\infty$, we have

$$\deg(P_k(D_{\langle p \rangle}) f \cdot f) \le 2 \deg(f) - \deg(P_k) ,$$

since

$$P_k(D_{\langle p \rangle}) f \cdot f = \sum_J \alpha_J \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_M}}{\partial x_M^{j_M}} f(x) \frac{\partial^{n_1 - j_1}}{\partial x_1^{n_1 - j_1}} \cdots \frac{\partial^{n_M - j_M}}{\partial x_M^{n_M - j_M}} f(x)$$

for some real constants α_J with $J = (j_1, \ldots, j_M)$. As a corollary, for a linear or quadratic function f (i.e., $\deg(f) \leq 2$), we have

$$P_k(D_{\langle p \rangle}) f \cdot f = 0\,, \tag{16}$$

when $\deg(P_k) \geq 5$.

In general, if f is a polynomial solution to the generalized bilinear equation (15), then the coefficients of f satisfy a group of nonlinear algebraic equations. For example, if $f(x,t) = ax^2 + 2bxt + ct^2 + dx + et + g$ is a solution of the bilinear KdV equation (10), then we have

$$ab = ac = ae = bc = cd = 12a^2 + 2bg - de = 0\,, \tag{17}$$

which leads to the following three classes of solutions:

(i) $f(x,t) = 2bxt + dx + et + de/(2b)$,
(ii) $f(x,t) = ct^2 + et + g$,
(iii) $f(x,t) = dx + g$.

For the bilinear KdV equation with $P = P(t,x) = xt + x^4$, we have

$$P(D) f \cdot f = 2(f_{xt} f - f_x f_t + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2) = 0\,,$$

when $p = 2$, and

$$P(D_{\langle 3 \rangle}) f \cdot f = 2(f_{xt} f - f_x f_t + 3 f_{xx}^2) = 0\,,$$

when $p = 3$. Therefore, the corresponding bilinear differential equations depend on the value of p. However, if f is quadratic, then we can easily find that

$$D_{p,i}^n f \cdot f = D_i^n f \cdot f, \quad n \geq 1\,, \tag{18}$$

where $1 \leq i \leq M$ and $p \geq 2$, and thus, the same quadratic function f can solve all generalized bilinear differential equations with different values of $p \geq 2$. We list this result as follows.

Theorem 1. *The generalized bilinear equations* (15) *with a given same polynomial* P *but different integers* $p \geq 2$ *possess the same set of quadratic function solutions.*

Let us denote a general quadratic function by

$$f(x) = x^T A x - 2b^T x + c\,, \tag{19}$$

where $A = A^T \in \mathbb{R}^{M \times M}$, $b \in \mathbb{R}^M$, and $c \in \mathbb{R}$, and write the polynomial $P(x)$ defining the generalized bilinear equation (15) as follows:

$$P(x) = \sum_{k=1}^{N} \sum_{i_1, \ldots, i_k = 1}^{M} p_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k}\,, \tag{20}$$

where $N = \deg(P) \geq 1$ is an integer, and $p_{i_1 \cdots i_k}, 1 \leq i_1, \ldots, i_k \leq M, 1 \leq k \leq N$, are real constants.

It has been proved[24] that a quadratic function f defined by (19) is positive everywhere in $\mathbb{R}^M$, i.e., $f(x) > 0$, $\forall x \in \mathbb{R}^M$, if and only if A is positive semi-definite, $b \in \text{range}(A)$ and $c - b^T A^+ b > 0$, where A^+ being the Moore–Penrose pseudoinverse of A.

Noting the properties in (16) and (18), we can see that only the coefficients p_{ij} and p_{ijkl} take effect in computing quadratic function solutions. Necessary and sufficient conditions on quadratic function solutions to Hirota bilinear equations have been presented in the previous paper[24] indeed. Based on Theorem 1, we can have the same criterion on quadratic function solutions to generalized bilinear equations stated below.

Theorem 2. *Let $A = (a_{ij})_{M\times M} \in \mathbb{R}^{M\times M}$ be symmetric and positive semi-definite, $b \in \mathbb{R}^M$ and $c \in \mathbb{R}$. The quadratic function f defined by* (19) *solves the generalized bilinear equation* (15) *with $P(x)$ defined by* (20) *if and only if*

$$2\sum_{i,j,k,l=1}^{M} p_{ijkl}(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}) + d\sum_{i,j=1}^{M} p_{ij}a_{ij} = 0 \tag{21}$$

and

$$\sum_{i,j=1}^{M} p_{ij}(a_{ij}A - A_iA_j^T - A_jA_i^T) = 0\,, \tag{22}$$

where A_i denotes the ith column vector of the symmetric matrix A for $1 \le i \le M$, and $d = c - b^T A^+ b$.

We point out that for distinct p, the generalized bilinear equations by (15) may have different polynomial solutions of higher order than two. For example, any C^3-differentiable function is a solution to the equation $D_x^3 f\cdot f = 0$. But if $f = f(x,t) = x^4 + t^2$, based on (14), we can have

$$D_{3,x}^3 f\cdot f = 48x(x^4+t^2) \neq 0\,,$$

which means that this quartic function f does not solve $D_{3,x}^3 f\cdot f = 0$.

3. Lump-Type Solutions to Two Classes of Nonlinear Differential Equations

In general, it is difficult to find rational function solutions to nonlinear differential equations. But using Mathematical software such as Maple, we can find polynomial solutions to generalized bilinear differential equations.

In this section, we will try to search for positive quadratic or quartic polynomial solutions to generalized bilinear equations. From those polynomial solutions f, we will be able to construct lump-type solutions to nonlinear differential equations, via the transformation of dependent variables $u = (2\ln f)_x$.

3.1. *First class of nonlinear equations*

Let us begin with the following polynomial [see formula (18) in Ref. 2]:

$$P = c_1x^5 + c_2x^3y + c_3x^2z + c_4xt + c_5yz\,,$$

where the coefficients c_i, $1 \le i \le 5$, are free real constants. The associated generalized bilinear differential equation with $p = 3$ reads [see (19) in Ref. 2]:

$$\begin{aligned} P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f &= 2c_1(f_{xxxxx}f - 5f_{xxxx}f_x + 10f_{xxx}f_{xx}) + 6c_2f_{xx}f_{xy} \\ &\quad + 2c_3f_{xxz}f + 2c_4(f_{xt}f - f_xf_t) \\ &\quad + 2c_5(f_{yz}f - f_yf_z) = 0\,. \end{aligned} \tag{23}$$

Taking $u = 2(\ln f)_x$ generates the corresponding nonlinear differential equation:

$$\begin{aligned} &\frac{\partial}{\partial x}\frac{P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f}{f^2} \\ &\quad = c_1\left(\frac{15}{2}u_x^3 + \frac{5}{2}u^3u_{xx} + \frac{15}{8}u^4u_x + 10u_xu_{xxx} + \frac{15}{2}u^2u_x^2\right. \\ &\qquad \left. + 15uu_xu_{xx} + 10u_{xx}^2 + u_{xxxxx}\right) \\ &\qquad + c_2\left[\frac{3}{8}u^3u_y + \frac{3}{2}u_xu_{xy} + \frac{3}{4}u^2u_{xy} + \frac{3}{2}u_{xx}u_y\right. \\ &\qquad \left. + \frac{9}{4}uu_xu_y + \frac{3}{8}(3u^2u_x + 2uu_{xx} + 2u_x^2)v\right] \\ &\qquad + c_3\left[uu_{xz} + u_{xxz} + \frac{3}{2}u_xu_z + \frac{1}{4}u^2u_z + \frac{1}{2}(u_{xx} + uu_x)w\right] + c_4u_{xt} + c_5u_{yz} = 0, \end{aligned} \tag{24}$$

where $u_y = v_x$ and $u_z = w_x$. Therefore, if f solves the generalized bilinear equation (23), then $u = 2(\ln f)_x$ solves the nonlinear differential equation (24).

3.1.1. *Quadratic function solutions*

Let us first consider quadratic function solutions to the generalized bilinear equation (23), which involve a sum of two squares. Based on the discussion in Sec. 2, we know that such solutions have nothing to do with c_1 and c_3. Therefore, the coefficients c_1 and c_3 will be arbitrary real constants. Three cases of such solutions by symbolic computation with Maple are displayed as follows.

(1) When $c_4 \ne 0$, but c_2 and c_5 are arbitrary, we have

$$\begin{aligned} f &= \left(\frac{a_4a_7a_8c_5}{{a_2}^2c_4}t + a_2x - \frac{a_7a_8}{a_2}y + a_4z + a_5\right)^2 \\ &\quad + \left(-\frac{a_4a_8c_5}{a_2c_4}t + a_7x + a_8y + \frac{a_4a_7}{a_2}z + a_{10}\right)^2 + a_{11}\,, \end{aligned}$$

where a_2, a_4, a_5, a_7, a_8, a_{10} and a_{11} are arbitrary real constants satisfying $a_2 \neq 0$ and $a_{11} > 0$.

(2) When $c_2 c_4 \neq 0$, but c_5 is arbitrary, we have

$$f = \left[\frac{a_3({a_4}^2 a_{11} c_5 - 3{a_7}^3 a_9 c_2) c_5}{3{a_7}^4 c_2 c_4} t + a_3 y + a_4 z + a_5\right]^2 + \left[\frac{a_3 a_4 \left(3{a_7}^3 c_2 + a_9 a_{11} c_5\right) c_5}{3{a_7}^4 c_2 c_4} t + a_7 x - \frac{a_3 a_4 a_{11} c_5}{3{a_7}^3 c_2} y + a_9 z + a_{10}\right]^2 + a_{11}\,,$$

where $a_3, a_4, a_5, a_7, a_9, a_{10}$ and a_{11} are arbitrary real constants satisfying $a_7 \neq 0$ and $a_{11} > 0$.

(3) When $c_4 c_5 \neq 0$, but c_2 is arbitrary, we have

$$f = [-(a_2 a_3 a_4 - a_2 a_8 a_9 + a_3 a_7 a_9 + a_4 a_7 a_8) d_1 t + a_2 x + a_3 y + a_4 z + a_5]^2 + [-(a_2 a_3 a_9 + a_2 a_4 a_8 - a_3 a_4 a_7 + a_7 a_8 a_9) d_1 t + a_7 x + a_8 y + a_9 z + a_{10}]^2 + d_2$$

with

$$d_1 = \frac{c_5}{({a_2}^2 + {a_7}^2) c_4}, \quad d_2 = -\frac{3({a_2}^2 + {a_7}^2)^2 (a_2 a_3 + a_7 a_8) c_2}{(a_2 a_9 - a_4 a_7)(a_2 a_8 - a_3 a_7) c_5}\,,$$

where a_i, $i = 2, \ldots, 5, 7, \ldots, 10$ are arbitrary real constants satisfying $a_2 a_9 - a_4 a_7 \neq 0$ and $a_2 a_8 - a_3 a_7 \neq 0$.

3.1.2. *Quartic function solutions*

Let us now consider quartic function solutions to the generalized bilinear equation (23). A direct symbolic computation with Maple tells us seven classes of positive quartic function solutions.

(1) Solutions independent of y:

$$f = (a_2 x + a_4 z + a_5)^2 + (a_7 x + a_9 z + a_{10})^2 + (a_{14} z + a_{15})^4 + a_{16}\,,$$

where a_2, a_4, a_5, a_7, a_9, a_{10}, a_{14}, a_{15} and $a_{16} > 0$ are arbitrary real constants.

(2) Solutions independent of z:

$$f = \left(-\frac{a_7 a_8}{a_3} x + a_3 y + a_5\right)^2 + (a_7 x + a_8 y + a_{10})^2 + (a_{13} y + a_{15})^4 + a_{16}\,,$$

where a_3, a_5, a_7, a_8, a_{10}, a_{13}, a_{15} and $a_{16} > 0$ are arbitrary real constants.

(3) Solutions independent of x and y:

$$f = (a_1 t + a_4 z + a_5)^2 + (a_6 t + a_9 z + a_{10})^2 + (a_{11} t + a_{14} z + a_{15})^4 + a_{16}\,,$$

where a_1, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{14}, a_{15} and $a_{16} > 0$ are arbitrary real constants.

(4) Solutions independent of x and z:

$$f = (a_1 t + a_3 y + a_5)^2 + (a_6 t + a_8 y + a_{10})^2 + (a_{11} t + a_{13} y + a_{15})^4 + a_{16}\,,$$

where a_1, a_3, a_5, a_6, a_8, a_{10}, a_{11}, a_{13}, a_{15} and $a_{16} > 0$ are arbitrary real constants.

(5) When $c_5 \neq 0$, but c_k, $1 \le k \le 4$, are arbitrary, we have

$$f = \left(-\frac{a_7a_8a_{11}}{a_2a_{13}}t + a_2x - \frac{a_7a_8}{a_2}y - \frac{a_2a_{11}c_4}{a_{13}c_5}z + a_5\right)^2 + \left(\frac{a_8a_{11}t}{a_{13}} + a_7x + a_8y - \frac{a_7a_{11}c_4}{a_{13}c_5}z + a_{10}\right)^2 + (a_{11}t + a_{13}y + a_{15})^4 + a_{16}\,,$$

where $a_2a_{13} \neq 0$, $a_{16} > 0$ and all other involved parameters are arbitrary real constants.

(6) When $c_4 \neq 0$, but c_k, $1 \le k \le 3$, we have

$$f = \left(-\frac{a_4a_{13}c_5}{a_{11}c_4}x + a_4z + a_5\right)^2 + (a_{11}t + a_{13}y + a_{15})^4 + a_{16}\,,$$

where $a_{11} \neq 0$, $a_{16} > 0$ and all other involved parameters are arbitrary real constants.

(7) When $c_4 \neq 0$, but c_k, $1 \le k \le 3$, we have

$$f = \left(-\frac{a_3a_9c_5}{a_7c_4}t + a_3y + a_5\right)^2 + (a_7x + a_9z + a_{10})^2 + \left(-\frac{a_9a_{13}c_5}{a_7c_4}t + a_{13}y + a_{15}\right)^4 + a_{16}\,,$$

where $a_7 \neq 0$, $a_{16} > 0$ and all other involved parameters are arbitrary real constants.

3.1.3. *Discussions*

Lump solutions are rationally localized in all directions in the space. For the exact solutions we discussed above, this characteristic property equivalently requires

$$\lim_{x^2+y^2+z^2\to\infty} u(x,y,z,t) = 0\,, \quad \forall t \in \mathbb{R}\,,$$

where $u = 2(\ln f)_x$, and obviously, a sufficient condition for u to be a lump solution is

$$\lim_{x^2+y^2+z^2\to\infty} f(x,y,z,t) = \infty, \quad \forall t \in \mathbb{R}\,.$$

We point out that all the solutions presented above do not satisfy this criterion, but they are rationally localized in many directions in the space and thus, we call them lump-type solutions.

We consider a special case of the parameters and coefficients for the class of solutions in Sec. 3.1.2 (5). Choose $a_2 = a_{11} = a_{13} = a_{16} = 1$, $a_5 = a_{10} = a_{15} = 0$, $a_7 = -1$, $a_8 = 2$ and $c_4 = c_5 = 2$. Then, we have the following positive quartic function solution to the generalized bilinear Eq. (23):

$$f = (2t + x + 2y - z)^2 + (2t - x + 2y + z)^2 + (t + y)^4 + 1\,,$$

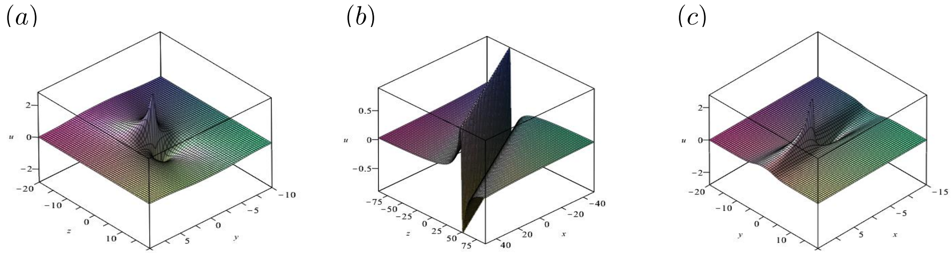

Fig. 1. Plots of (25) at $t = 0$ with (a) $x = 0$, (b) $y = -1$ and (c) $z = 1$.

and the corresponding lump-type solution to the nonlinear differential equation (24):

$$u = 2(\ln f)_x = \frac{8(x-z)}{(2t+x+2y-z)^2 + (2t-x+2y+z)^2 + (t+y)^4 + 1}\,. \qquad (25)$$

Figure 1 shows three $3d$ plots of this lump-type solution at $t = 0$ with $x = 0$, $y = -1$ and $z = 1$, respectively.

3.2. *Second class of nonlinear equations*

Let us now begin with the polynomial [see (29) of Ref. 2]:

$$P = c_1x^2 + c_2x^3y + c_3x^4y^2 + c_4yt\,,$$

where the coefficients c_i, $1 \le i \le 4$, are free real constants. The associated generalized bilinear differential equation with $p = 3$ [see (34) in Ref. 2] reads:

$$\begin{aligned} P(D_{3,x}, D_{3,y}, D_{3,t})f\cdot f &= 2c_1(f_{xx}f - f_x^2) + 6c_2f_{xx}f_{xy} + 2c_3(f_{xxxxyy}f \\ &\quad + 4f_{xxx}f_{xyy} + 6f_{xxy}^2) + 2c_4(f_{yt}f - f_yf_t) = 0\,. \end{aligned} \qquad (26)$$

Taking $u = (2\ln f)_x$ generates the corresponding nonlinear differential equation

$$\begin{aligned} &\frac{\partial}{\partial x}\frac{P(D_{3,x}, D_{3,y}, D_{3,t})f\cdot f}{f^2} \\ &= c_1u_{xx} + c_2\left[\frac{3}{8}u^3u_y + \frac{3}{2}u_xu_{xy} + \frac{3}{4}u^2u_{xy} + \frac{3}{2}u_{xx}u_y + \frac{3}{8}(3u^2u_x + 2uu_{xx} + 2u_x^2)w\right] \\ &\quad + c_3\left[\frac{5}{16}u^4u_{yy} + 12uu_{xy}^2 + 5u_yu_{xxxy} + \frac{27}{4}u_x^2u_{yy} + 16u_{xy}u_{xxy} + 5u_xu_{xxyy} + \frac{5}{2}u^3u_y^2\right. \\ &\quad + 2uu_{xxxyy} + u^3u_{xyy} + 7u_{xx}u_{xyy} + \frac{3}{2}u^2u_{xxyy} + \frac{9}{2}u_{xxx}u_{yy} + 7u_{xx}u_y^2 + 14uu_{xxy}u_y \\ &\quad + 9uu_xu_{xyy} + \frac{21}{4}u^2u_xu_{yy} + 24u_xu_yu_{xy} + 12u^2u_{xy}u_y + 8uu_{xx}u_{yy} + 18uu_xu_y^2 \\ &\quad + u_{xxxxyy} + \Big(u_{xxxxy} + 8u_xu_{xxy} + 2uu_{xxxy} + 3u^2u_{xxy} + \frac{5}{2}u^3u_{xy} + 15uu_xu_{xy} \end{aligned}$$

$$+\frac{9}{2}u_{xxx}u_y+10u_{xx}u_{xy}+2uu_{xxx}+\frac{11}{16}u^4u_y+11uu_yu_{xx}+\frac{45}{4}u^2u_xu_y+\frac{45}{4}u_x^2u_y\Big)\,w$$
$$+\left(\frac{1}{2}u_{xxxx}+\frac{9}{4}u^2u_{xx}+2uu_{xxx}+\frac{7}{2}u_xu_{xx}+\frac{5}{4}u^3u_x+\frac{9}{2}uu_x^2\right)w_y$$
$$+\left(\frac{1}{4}u_{xxxx}+\frac{15}{8}u^2u_{xx}+\frac{13}{4}u_xu_{xx}+uu_{xxx}+\frac{11}{8}u^3u_x+\frac{15}{4}uu_x^2\right)w^2\Big]+c_4u_{yt}=0\,, \tag{27}$$

where $u_y = w_x$.

In what follows, we will try to search for positive quadratic and quartic solutions to the generalized bilinear equation (26) by symbolic computations with Maple. The resulting polynomial solutions will yield lump-type and lump solutions to the nonlinear differential equation (27).

3.2.1. *Quadratic function solutions*

A direct symbolic computation tells the following four classes of positive quadratic function solutions to the generalized bilinear equation (26).

(1) When $c_1c_4 \neq 0$, but c_1 and c_2 are arbitrary, we have

$$f=\left[-\frac{({a_1}^2a_2+2a_1a_4a_5-a_2{a_4}^2)c_1}{({a_2}^2+{a_5}^2)c_4}t+a_1x+a_2y+a_3\right]^2+\left[\frac{({a_1}^2a_5-2a_1a_2a_4-{a_4}^2a_5)c_1}{({a_2}^2+{a_5}^2)c_4}t+a_4x+a_5y+a_6\right]^2-\frac{3(a_1a_2+a_4a_5)({a_1}^2+{a_4}^2)({a_2}^2+{a_5}^2)c_2}{(a_1a_5-a_2a_4)^2c_1}\,,$$

where a_i, $1 \le i \le 6$, are arbitrary real constants satisfying $a_1a_5 - a_2a_4 \neq 0$ and $(a_1a_2 + a_4a_5)c_1c_2 < 0$.

(2) When $c_1c_4 \neq 0$, but c_1 and c_2 are arbitrary, we have

$$f=\left[\frac{({a_4}^2{a_6}^2-{a_1}^2{a_3}^2)c_1}{a_2({a_3}^2+{a_6}^2)c_4}t+a_1x+a_2y+a_3\right]^2+\left[\frac{(a_4a_6-a_1a_3)(a_3a_4-a_1a_6)c_1}{a_2({a_3}^2+{a_6}^2)c_4}t+a_4x+\frac{a_2(a_3a_4+a_1a_6)}{a_1a_3-a_4a_6}y+a_6\right]^2-\frac{3a_2a_3({a_3}^2+{a_6}^2)({a_1}^2+{a_4}^2)c_2}{a_6^2(a_1a_3-a_4a_6)c_1}\,,$$

where a_i, $1 \le i \le 4$ and a_6 are arbitrary real constants satisfying $a_2 \neq 0$, $a_6 \neq 0$, $a_1a_3 - a_4a_6 \neq 0$ and $a_2a_3(a_1a_3 - a_4a_6)c_1c_2 < 0$.

(3) When $c_1c_4 \neq 0$, but c_2 and c_3 are arbitrary, we have

$$f = [a_7t + a_1x + a_2y + a_3]^2 + [a_8t + a_4x + a_5y + a_6]^2 + a_9$$

with

$$a_7=\frac{(a_2{a_4}^2-2a_1a_4a_5-{a_1}^2a_2)c_1}{({a_2}^2+{a_5}^2)c_4}\,,$$

$$a_8 = \frac{({a_1}^2 a_5 - 2a_1a_2a_4 - {a_4}^2 a_5)c_1}{({a_2}^2 + {a_5}^2)c_4},$$

$$a_9 = -\frac{3(a_1a_2 + a_4a_5)({a_1}^2 + {a_4}^2)({a_3}^2 + {a_6}^2)c_2}{(a_1a_5 - a_2a_4)^2 c_1},$$

where a_i, $1 \le i \le 6$, are arbitrary real constants satisfying $a_1a_5 - a_2a_4 \neq 0$ and $a_9 > 0$.

(4) When $c_1c_4 \neq 0$, but c_2 and c_3 are arbitrary, we have

$$f = (a_7t + a_1x + a_2y + a_3)^2 + (a_8t + a_4x + a_5y + a_6)^2 + a_9$$

with

$$a_5 = \frac{a_2(a_1a_6 + a_3a_4)}{a_1a_3 - a_4a_6},$$

$$a_7 = -\frac{({a_1}^2{a_3}^2 - {a_4}^2{a_6}^2)c_1}{a_2({a_3}^2 + {a_6}^2)c_4},$$

$$a_8 = \frac{(a_1a_3 - a_4a_6)(a_1a_6 - a_3a_4)}{a_2({a_3}^2 + {a_6}^2)c_4},$$

$$a_9 = -\frac{3({a_1}^2 + {a_4}^2)({a_3}^2 + {a_6}^2)a_2a_3c_2}{(a_1a_3 - a_4a_6){a_6}^2 c_1},$$

where a_i, $1 \le i \le 4$ and a_6 are arbitrary real constants satisfying $a_2 \neq 0$, $a_1a_3 - a_4a_6 \neq 0$, and $a_9 > 0$.

3.2.2. *Quartic function solutions*

A direct symbolic computation with f involving a sum of three squares leads to the following three classess of positive quartic function solutions to the generalized bilinear equation (26).

(1) Case I involving a sum of three squares:

$$f = (a_1y + a_2)^2 + (a_3y + a_4)^2 + \left[a_5ty + \frac{(a_1a_2 + a_3a_4)a_5}{{a_1}^2 + {a_3}^2}t\right]^2 - \frac{(a_1a_4 - a_2a_3)^2}{{a_1}^2 + {a_3}^2},$$

where a_i, $1 \le i \le 5$, are arbitrary real constants satisfying ${a_1}^2 + {a_3}^2 \neq 0$.

(2) Case II involving a sum of three squares:

$$f = \left(a_1 - \frac{a_2a_3}{a_1}t\right)^2 + (a_2t + a_3)^2 + (a_4t + a_5ty)^2 - {a_1}^2 - {a_3}^2,$$

where $a_1 \neq 0$ and a_i, $2 \le i \le 5$, are arbitrary real constants.

(3) Case III involving a sum of three squares:

$$f = \left(a_1 - \frac{a_2a_3}{a_1}t + \frac{a_1a_5}{a_4}y\right)^2 + \left(a_2t + \frac{a_3a_5}{a_4}y + a_3\right)^2 + (a_4t + a_5ty)^2 + \frac{{a_2}^2({a_1}^2 + {a_3}^2)^2}{{a_1}^2{a_4}^2},$$

where a_i, $1 \le i \le 5$, are arbitrary real constants satisfying $a_1a_4 \neq 0$.

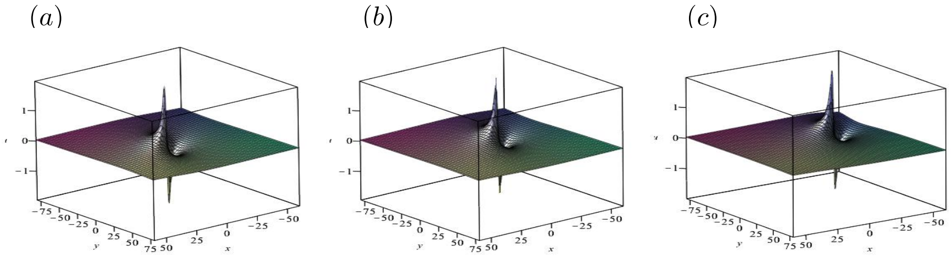

Fig. 2. Plots of (28) at (a) $t = 0$, (b) $t = 15$ and (c) $t = 30$.

3.2.3. *Discussions*

We can generate lump solutions from the presented quadratic function solutions to the generalized bilinear equation (26) in Sec. 3.2.1. But the quartic function solutions in Sec. 3.2.2 are all independent of the spatial variable x. Therefore, when y is fixed and x goes to ∞, f will not tend to ∞. This implies that the presented quartic function solutions to the generalized bilinear equation (26) will not produce any lump solution to the nonlinear differential equation (27).

Let us now present a special class of lump solutions from the first class of quadratic function solutions in Sec. 3.2.1. To the end, we specify $a_2 = a_4 = 1$, $a_5 = 0$, $a_1 = -\text{sgn}(c_1c_2)$. Then, we have the positive quadratic function solution to (26):

$$f(x,y,t) = (a_1x + y + a_3)^2 + \left(-\frac{2a_1c_1}{c_4}t + x + a_6\right)^2 + 6\left|\frac{c_2}{c_1}\right|$$

and the lump solution to (27):

$$u(x,y,t) = \frac{4\left[-\frac{2a_1c_1}{c_4}t + ({a_1}^2+1)x + a_1y + a_1a_3 + a_6\right]}{(a_1x+y+a_3)^2 + \left(\frac{-2a_1c_1}{c_4}t + x + a_6\right)^2 + 6\left|\frac{c_2}{c_1}\right|}.$$

Particularly, taking $a_1 = -1$, $a_3 = 0$, $a_6 = -1$ and $c_1 = 1$, $c_2 = 1/3$, $c_4 = 2$ leads to the following lump solution:

$$u(x,y,t) = \frac{4(2x - y + t - 1)}{(-x+y)^2 + (t+x-1)^2 + 2}. \tag{28}$$

Figure 2 shows three 3D plots of this solution at $t = 1, 15, 30$, respectively. The plots depict that the lump by (28) does not change much while it travels.

4. Concluding Remarks

We have analyzed quadratic function solutions to generalized bilinear equations, and carried out a search for positive quadratic and quartic function solutions to two classes of generalized bilinear equations, generated from the two specific polynomials. The resulting polynomial solutions yield lump-type solutions of finite energy to the corresponding nonlinear differential equations, derived from the considered generalized bilinear equations. It is expected that such a study will help us recognize the characteristics of generalized bilinear equations and nonlinear differential equations.

High-order polynomial solutions to generalized bilinear equations is an interesting problem. It will be important to see if there is any nonlinear superposition formula for generating lump solutions in terms of higher-order polynomials. Is there any combinatorial relation between higher-order polynomial solutions and generalized bilinear equations? The second interesting problem is how to construct lump solutions to discrete integrable equations. Rational function solutions to the Toda and 2-dimensional Toda lattice equations are successfully presented and expressed in terms of Casoratian determinant.[23,29] The third problem is to see if it is possible to classify lump solutions from a determinant point of view. Can lump solutions be written in Wronskian, Casoratian or Pfaffian form?

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